

## COUNTING HYPERBOLIC COMPONENTS

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ABSTRACT. We give formulas for the numbers of type II and type IV hyperbolic components in the space of quadratic rational maps, for all fixed periods of attractive cycles.

## 1. INTRODUCTION

We present here a calculation of numbers of hyperbolic components in the moduli space of quadratic rational maps with marked critical points, regarded as dynamical systems acting on the Riemann sphere. Hyperbolic components of quadratic rational maps come in four types.

**Type I.:** There is only one such component. The Julia set is totally disconnected, so that the Fatou set is connected and multiply connected. Both critical points are in the Fatou set.

**Type II.:** There is one periodic orbit of Fatou components, with both critical components in this periodic orbit of components, but in distinct components in the orbit. The period of the orbit is  $> 1$ . For any fixed period, there are just finitely many type II hyperbolic components for this period: one for period two, and two of period 3, for example.

**Type III.:** For this type also, there is just one periodic orbit of Fatou components, of some period  $> 1$ . This orbit of components contains just one of the critical points, in a component  $U$ , say, of the Fatou set. The other critical point, and hence also the corresponding critical value, are contained in the backward orbit of the periodic orbit of  $U$ . There are two numbers associated to a type III component: the *period*, that is, the period  $n$  of the unique periodic orbit of Fatou components, and the *preperiod*  $m$  of the nonperiodic component  $V$  of the Fatou set with contains the second critical value, that is, the least integer  $m$  such that the  $m$ 'th forward image of  $V$  is periodic. There are infinitely many type III components of any fixed period, but only finitely many if we fix the period and preperiod.

**Type IV.:** For this type, there are two disjoint periodic orbits of Fatou components, one containing each of the critical points. There are infinitely many type IV components if we fix the period of one orbit, but only finitely many if we fix the periods of both cycles.

We will count numbers of hyperbolic components of types II and IV in the space  $\mathcal{M}_2^{cm}$  of critically marked quadratic rational maps. The elements of  $\mathcal{M}_2^{cm}$

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are conjugacy classes of triples  $(f, \omega_1, \omega_2)$  where  $f$  is a quadratic rational map with critical points at  $\omega_1$  and  $\omega_2$ . More precisely,  $(f, \omega_1, \omega_2)$  and  $(g, \omega'_1, \omega'_2)$  are conjugate if there exists a Möbius transformation  $\gamma$  such that  $\gamma \circ f = g \circ \gamma$  and  $\gamma(\omega_j) = \omega'_j$  for  $j = 1, 2$ . We will denote the conjugacy class of  $(f, \omega_1, \omega_2)$  by  $[f, \omega_1, \omega_2]$ . The space  $\mathcal{M}_2^{cm}$  is naturally endowed with a two complex dimensional orbifold structure [7, Section 6]. More precisely,  $\mathcal{M}_2^{cm}$  may be identified with a hypersurface in  $\mathbb{C}^3$  with exactly one singular point [7, Lemma 6.1].

We start counting components of type IV. To state our results we will need to introduce the numbers  $\nu_q(n)$ . For  $q > 1$  and  $n \geq q$ , let  $\nu_q(n)$  be the number of hyperbolic components of period dividing  $n$  in a  $p/q$ -limb of the Mandelbrot set (e.g. for the definition of limbs see [3] or [8]).

More precisely, if

$$r \equiv n \pmod{q},$$

and  $0 \leq r < q$ , then

$$\nu_q(n) = \begin{cases} \frac{2^{n-1} - 2^{r-1}}{2^q - 1} & \text{if } q \nmid n \\ \frac{2^{n-1} - 2^{r-1}}{2^q - 1} + \frac{1}{2} & \text{if } q \mid n. \end{cases}$$

Denote by  $\phi(n)$  the Euler Phi function of  $n$  (i.e., the number of integers  $1 \leq k \leq n$  which are relatively prime to  $n$ ).

We shall prove the following theorem.

**Theorem 1.1.** *Consider integers  $m \geq n \geq 1$ . Let  $\eta_{IV}(n, m)$  be the number of type IV components with one attracting cycle of period dividing  $n$ , and other attracting cycle of period dividing  $m$ . Let  $\eta_{II}(\text{mcd}(n, m))$  be the number of type II components with an attracting cycle of period dividing  $\text{mcd}(n, m)$ . Then*

$$(1.1) \quad \eta_{IV}(n, m) = \frac{1}{3} (5 \cdot 2^{n+m-3} + 2^{n-2} + 2^{m-2}) - \frac{1}{2} \sum_{2 \leq q \leq n} \phi(q) \nu_q(n) \nu_q(m)$$

$$- \eta_{II}(\text{mcd}(n, m)) + \frac{1}{6} ((-1)^n + (-1)^m + (-1)^{n+m})$$

In particular, given  $n \geq 1$ ,

$$\eta_{IV}(n, m) = \left( \frac{5}{3} \cdot 2^{n-3} + \frac{1}{12} - \frac{1}{4} \sum_{2 \leq q \leq n} \frac{\phi(q) \nu_q(n)}{2^q - 1} \right) \cdot 2^m + \varepsilon_n(m),$$

where  $\varepsilon_n(m)$  is a bounded function of  $m$ . More precisely,

$$|\varepsilon_n(m)| \leq 2^n + 2^{2 \text{mcd}(n, m)}.$$

**Remark 1.** The number of type IV hyperbolic components with one cycle of period exactly  $n$  and the other of period dividing  $m$  is:

$$\begin{aligned}
2^{m-1} & \quad \text{if } n = 1, \\
\frac{1}{3} \cdot 2^m + O(1) & \quad \text{if } n = 2, \\
\frac{23}{21} \cdot 2^m + O(1) & \quad \text{if } n = 3, \\
\frac{78}{35} \cdot 2^m + O(1) & \quad \text{if } n = 4, \\
\frac{6103}{1085} \cdot 2^m + O(1) & \quad \text{if } n = 5, \\
\frac{202371}{19530} \cdot 2^m + O(1) & \quad \text{if } n = 6, \\
\frac{29316701}{1240155} \cdot 2^m + O(1) & \quad \text{if } n = 7.
\end{aligned}$$

2. The analogue of the result for the type III components is proved in the case when  $n = 3$  in [10], in a very simple-minded way, where it is pointed out that the type IV calculation can be done similarly. The simple-minded calculation for  $n = 3$  agrees with the result of the theorem.

We also consider the case of type II components. It is easier to write the formula in terms of the number  $\eta'_{II}(m)$  of type II hyperbolic components of period exactly  $m$ , so that

$$\eta_{II}(m) = \sum_{d|m} \eta'_{II}(d)$$

As before,  $\phi$  denotes the Euler  $\phi$ -function, and we define

$$\nu'_q(j+1) = \begin{cases} 0 & \text{if } j < q, \\ \frac{2^j - 2^r}{2^q - 1} & \text{if } j \geq q, \quad r = j \bmod q, \quad 0 \leq r < q-2, \\ 1 + \frac{2^j - 2^{q-1}}{2^q - 1} & \text{if } j \geq q, \quad q-1 = j \bmod q. \end{cases}$$

**Theorem 1.2.** For  $m \geq 3$ ,

$$\begin{aligned}
(1.2) \quad \sum_{d|m, d \geq 3} \frac{m}{d} \eta'_{II}(d) &= \frac{7}{36} m 2^m - \frac{37}{108} 2^m - \frac{m}{4} - (-1)^m \frac{5}{36} m + \frac{1}{2} + (-1)^m \frac{5}{54} \\
&\quad - \frac{1}{2} \sum_{3 \leq q, q < j < m-q} \phi(q) \nu'_q(j) \nu'_q(m-j).
\end{aligned}$$

**Remark** We have

$$\eta'_{II}(m) = \begin{cases} 1 & \text{if } m = 2, \\ 2 & \text{if } m = 3, \\ 6 & \text{if } m = 4, \\ 20 & \text{if } m = 5, \\ 47 & \text{if } m = 6, \\ 130 & \text{if } m = 7, \\ 295 & \text{if } m = 8 \end{cases}$$

There is a (negative) contribution from  $\phi(q)\nu'_q(j)\nu'_q(m-j)$  only for  $m \geq 8$ . When  $m = 8$  the contribution is only for  $j = 4$  and  $q = 3$ , and  $\nu'_3(4) = 1$ .

The proof of Theorem 1.1 regarding type IV components is contained in Section 2 and the proof of Theorem 1.2 regarding type II components is contained in Section 3. In the Appendix we include a discussion about the numbers  $\nu'_q(n)$  used in the later proof.

## 2. TYPE IV COMPONENTS

**2.1. Appropriate projective curves for type IV components.** According to [9] each hyperbolic component of type II or IV contains a unique postcritically finite quadratic rational map, called the *center* of the hyperbolic component. We consider the curves  $X_n$  and  $Y_m$  in  $\mathcal{M}_2^{cm}$  where  $\omega_1$  has period dividing  $n$  and  $\omega_2$  has period dividing  $m$ , respectively. Our aim is to compute the cardinality of  $X_n \cap Y_m$  since the maps in this intersection are in one to one correspondence with the center of hyperbolic components of type II or IV with a Fatou component of period dividing  $n$  containing  $\omega_1$  and a Fatou component of period dividing  $m$  containing  $\omega_2$ .

The curves  $X_n$  and  $Y_m$  are the union of *periodic curves*. That is, for  $p \geq 1$ , we let  $V_p$  (resp.  $W_p$ ) be the subset of  $\mathcal{M}_2^{cm}$  formed by all  $[f, \omega_1, \omega_2]$  such that  $\omega_1$  (resp.  $\omega_2$ ) is periodic of period exactly  $p$ . It follows that:

$$\begin{aligned} X_n &= \cup_{p|n} V_p, \\ Y_m &= \cup_{p|m} W_p. \end{aligned}$$

For our purpose it is convenient to work with the set  $\mathcal{R} = \mathcal{M}_2^{cm} \setminus X_2$  which may be parametrised as follows. For  $(c, d) \in \mathbb{C} \times \mathbb{C}^*$ , consider the quadratic rational map

$$f_{c,d} = 1 + \frac{c}{z} + \frac{d}{z^2}.$$

Then  $(c, d) \mapsto [f_{c,d}, 0, -2d/c]$  parametrises  $\mathcal{R} = \mathcal{M}_2^{cm} \setminus X_2$ . That is we identify  $\mathcal{R}$  with  $\mathbb{C} \times \mathbb{C}^*$ .

For us, it is also convenient to regard  $\mathcal{R}$  as a subset of  $\mathbb{CP}^2$ . That is, by adding the line  $d = 0$  and a projective line at infinity to  $\mathcal{R}$ , we obtain  $\mathbb{CP}^2$ , with a preferred affine plane  $\mathbb{C}^2$  parametrised by  $(c, d)$ . Thus, we will regard  $\mathcal{R}$  both as a subset of  $\mathcal{M}_2^{cm}$  and of  $\mathbb{CP}^2$  according to convenience.

In order to be precise we let  $\mathcal{X}_n = X_n \cap \mathcal{R} \subset \mathbb{CP}^2$  and  $\mathcal{Y}_m = Y_m \cap \mathcal{R} \subset \mathbb{CP}^2$ . Similarly, let  $\mathcal{V}_n = V_n \cap \mathcal{R}$  and  $\mathcal{W}_m = W_m \cap \mathcal{R}$ . Denote by  $\overline{\mathcal{X}_n}$  and  $\overline{\mathcal{Y}_m}$  their closure in  $\mathbb{CP}^2$ . It follows that  $\overline{\mathcal{X}_n}$ ,  $\overline{\mathcal{Y}_m}$ ,  $\overline{\mathcal{V}_n}$  and  $\overline{\mathcal{W}_m}$  are algebraic varieties. We recall that for  $n \geq 3$  and  $m \geq 1$ ,  $\mathcal{V}_n$  and  $\mathcal{W}_m$  are smooth, and their intersections are transverse. Smoothness actually holds everywhere on  $\mathcal{V}_n$  and  $\mathcal{W}_m$ , by an argument involving

the Measurable Riemann Mapping Theorem, for which we are unable to provide a published reference, but which is exactly the same as the argument used by Douady in [2] to show that hyperbolic components in the family of quadratic polynomials  $q_c(z) = z^2 + c$  ( $c \in \mathbb{C}$ ) are parametrised by the multiplier of the finite periodic attractive cycle. However, we only need to know smoothness – and transversality – at intersections between  $\mathcal{V}_n$  and  $\mathcal{W}_m$ . For this, it suffices to note that hyperbolic components of types II and IV are parametrised by spaces of Blaschke products (except for one exceptional type II component of period two). This is proved in [6, Theorem 4.1]: the proof works for rational maps in general, not just for spaces of polynomials. Alternatively, one can use the description of the parametrisation of these hyperbolic components given in the main theorem of [9]: although it is not explicitly said so, it is clear from the description given that the parametrisation is smooth, and, in fact, real analytic.

The proof of Theorem 1.1 relies on rewriting the formula as follows:

(2.1)

$$\begin{aligned}
\eta_{IV}(n, m) &= \frac{1}{36}(2^n - 3 - (-1)^n)(7 \cdot 2^m + 3 - (-1)^m) \\
&\quad - \frac{1}{2} \sum_{3 \leq q \leq n} \phi(q) \nu_q(n) \nu_q(m) \\
&\quad - \frac{1}{2} \left( \frac{2^n}{6} + \frac{(-1)^n}{3} \right) \left( \frac{2^m}{6} + \frac{(-1)^m}{3} \right) \\
&\quad + \frac{1}{2} \left( \frac{2^n}{6} + \frac{(-1)^n}{3} \right) \left( \frac{2^m}{6} + \frac{(-1)^m}{3} \right) \\
&\quad + \frac{(4 + (-1)^n)}{6} 2^m - \frac{(1 + (-1)^n)(-1)^m}{6} + \frac{(1 + (-1)^n)(1 + (-1)^m)}{4} \\
&\quad - \eta_{II}(\text{mcm}(n, m)).
\end{aligned}$$

The first two lines add up to the size of  $\mathcal{X}_n \cap \mathcal{Y}_m$ , for  $n \geq 3, m \geq 1$ . That is, the total number of type II and IV components with an attracting cycle of period  $\geq 3$  dividing  $n$  and an attracting cycle of period dividing  $m$  (maybe the same cycle). In order to compute the size of  $\mathcal{X}_n \cap \mathcal{Y}_m$  we will apply Bezout's Theorem. The first line is the product of the degrees of  $\overline{\mathcal{X}}_n$  and  $\overline{\mathcal{Y}}_m$  (see Subsection 2.2) and the second line is their intersection number outside  $\mathcal{R}$  (see Sections 2.3 and 2.4).

The third and fourth line cancel out. However, the third line is  $-(1/2)\phi(2)\nu_2(n)\nu_2(m)$  so that we may insert this number in the sum of the second line of (2.1).

The fifth line corresponds to  $\eta_{IV}(1, m)$  if  $n$  is odd, to  $\eta_{IV}(2, m)$  if  $n$  is even and  $m$  is odd, and to  $\eta_{IV}(2, m) + \eta_{II}(2)$  if  $n$  and  $m$  are even.

The last line is the necessary correction to only count type IV components.

## 2.2. The degrees of the curves.

**Lemma 2.1.** *The following statements hold:*

- For all  $n \geq 3$ , the degree of  $\overline{\mathcal{X}_n}$  is

$$\frac{1}{6}2^n - \frac{1}{2} - \frac{1}{6}(-1)^n.$$

- For all  $m \geq 1$ , the degree of  $\overline{\mathcal{Y}_m}$  is

$$\frac{7}{6}2^m + \frac{1}{2} - \frac{1}{6}(-1)^m.$$

- None of the points  $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$  belongs to  $\overline{\mathcal{X}_n}$ , for all  $n \geq 3$ .

*Proof.* Let

$$P_3(c, d) = 1 + c + d, \quad Q_3(c, d) = 1, \quad R_1(c, d) = 4d - c^2, \quad S_1(c, d) = d.$$

For  $n \geq 3$ , let

$$P_{n+1} = P_n^2 + cP_nQ_n + dQ_n^2, \quad Q_{n+1} = P_n^2.$$

For  $m \geq 1$ , let

$$R_{m+1} = R_m^2 + cR_mS_m + dS_m^2, \quad S_{m+1} = R_m^2.$$

Note that

$$f_{c,d}^n(0) = \frac{P_n(c, d)}{Q_n(c, d)}, \quad f_{c,d}^m(-2d/c) = \frac{R_m(c, d)}{S_m(c, d)}.$$

Thus,  $\mathcal{X}_n$  (resp.  $\mathcal{Y}_m$ ) is the set of all  $(c, d) \in \mathcal{R}$  such that  $P_n(c, d) = 0$  (resp.  $cR_m(c, d) + 2dS_m(c, d) = 0$ ). Denote by  $P_n^h(c, d, e)$  (resp.  $G_m^h(c, d, e)$ ) the homogeneous version of  $P_n(c, d)$  (resp.  $cR_m(c, d) + 2dS_m(c, d)$ ). Then  $\overline{\mathcal{X}_n}$  (resp.  $\overline{\mathcal{Y}_m}$ ) is the projective curve defined by  $P_n^h(c, d, e)$  (resp.  $G_m^h(c, d, e)$ ).

Recursively, it is easy to check that the constant term of  $P_n$  is 1, and that among the terms with maximal degree  $\deg(P_n)$ , both the monomials  $c^{\deg(P_n)}$  and  $d^{\deg(P_n)}$  always appear multiplied by appropriate (non-zero) coefficients. Thus, none of the points  $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$  belongs to  $\overline{\mathcal{X}_n}$ .

Using the recursive formulas above, a straightforward induction shows that:

$$\deg(P_n^h) = \frac{1}{6}2^n - \frac{1}{2} - \frac{1}{6}(-1)^n$$

and

$$\deg(G_m^h) = \frac{7}{6}2^m + \frac{1}{2} - \frac{1}{6}(-1)^m.$$

So to prove the lemma it is sufficient to establish that  $P_n^h$  generates the ideal of  $\overline{\mathcal{X}_n}$  and, similarly that  $G_m^h$  generates the ideal of  $\overline{\mathcal{Y}_m}$ .

The birational automorphism of  $\mathbb{CP}^2$  induced by interchanging the role of the critical points will allow to us to only check the above for  $P_n^h$ . More precisely, for all  $(c, d) \in \mathcal{R} \setminus \mathcal{Y}_2$ , let  $M_{c,d}$  be the Möbius transformation such that:

$$M_{c,d}(f_{c,d}^j(-2d/c)) = \begin{cases} 0 & \text{if } j = 0, \\ \infty & \text{if } j = 1, \\ 1 & \text{if } j = 2. \end{cases}$$

Then, there exists  $(c', d') \in \mathcal{R}$  such that

$$f_{(c',d')} = M_{c,d}^{-1} \circ f_{c,d} \circ M_{c,d}.$$

It follows that  $c'$  and  $d'$  are rational functions of  $(c, d)$ . Moreover,  $M_{c', d'} = M_{c, d}^{-1}$ . Therefore, the map  $(c, d) \mapsto (c', d')$  is a birational map  $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ . Furthermore,

$$\frac{G_m^h}{G} = \varphi^*(P_m^h) = P_m^h \circ \varphi,$$

where  $G = G_1^h$  if  $m$  is odd, and  $G = G_2^h$  if  $m$  is even. After checking that  $G_1^h$  and  $G_2^h$  are free of perfect square factors in  $\mathbb{C}[c, d, e]$ , it follows that  $G_m^h$  generates the ideal of  $\overline{\mathcal{Y}_m}$  if and only if  $P_m^h$  generates the ideal of  $\overline{\mathcal{X}_m}$  for all  $m \geq 3$ .

To show that  $P_n^h$  is square factor free, it is sufficient to show that the degree of  $\overline{\mathcal{X}_n}$  coincides with that of  $P_n$ . For this we count the intersections of  $\overline{\mathcal{X}_n}$  with the degree 3 curve  $\overline{\mathcal{Y}_1}$ . The curve  $\overline{\mathcal{Y}_1}$  is defined by the equation  $c^3 - 4dce - 8d^2e$ . Thus,  $\overline{\mathcal{Y}_1}$  intersects the line  $d = 0$  at  $[0 : 0 : 1]$  and the line  $e = 0$  at  $[0 : 1 : 0]$ . Therefore  $\overline{\mathcal{Y}_1} \cap \overline{\mathcal{X}_n}$  is contained in  $\mathcal{R}$ . Hence,  $\overline{\mathcal{Y}_1} \cap \overline{\mathcal{X}_n}$  consists of points of transverse intersection between  $\mathcal{Y}_1$  and  $\mathcal{X}_n$ . It follows that  $3 \deg(\overline{\mathcal{X}_n})$  coincides with the cardinality of  $\mathcal{Y}_1 \cap \mathcal{X}_n$ . Moreover,  $\{f_{c,d} \mid (c, d) \in \mathcal{Y}_1\}$  is, modulo change of coordinates, the quadratic family  $\{z^2 + v \mid v \in \mathbb{C}\}$ . Thus, points in  $\mathcal{Y}_1 \cap \mathcal{X}_n$  are in one to one correspondence with polynomials of the form  $z^2 + v$  such that  $z = 0$  is periodic of period  $q$  where  $3 \leq q \mid n$ . More precisely, the cardinality of  $\mathcal{Y}_1 \cap \mathcal{X}_n = 2^{n-1} - 1 - \delta$  where  $\delta = 1$  if  $n$  is odd, and  $\delta = 2$  if  $n$  is even. That is,

$$3 \deg(\overline{\mathcal{X}_n}) = 2^{n-1} - 1 - \frac{1 + (-1)^n}{2} = 3 \deg(P_n^h),$$

and the lemma follows.  $\square$

**Remark** From the proof above it follows that the birational map  $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$  has degree 7.

To prove Theorem 1.1 we need to compute the cardinality of the intersection

$$\mathcal{X}_n \cap \mathcal{Y}_m.$$

But by Bezout's theorem the number of intersections in  $\mathbb{CP}^2$  is simply the product of the degrees. So now we need to compute the number of intersections, with multiplicities, in  $\mathbb{CP}^2 \setminus \mathcal{R}$ . Counting the intersections in  $\mathbb{CP}^1 \setminus \mathcal{R}$  is divided into two parts. First we show that there are no intersections in the line  $[c : d : 0]$  and then we count the intersections in the line  $[c : 0 : 1]$ .

**2.3. No intersections at  $c = d = \infty$ .** Our aim now is to prove the following lemma.

**Lemma 2.2.** *For all  $n \geq 3$  and  $m \geq 1$ ,*

$$\overline{\mathcal{X}_n} \cap \overline{\mathcal{Y}_m} \cap \{x \in \mathbb{CP}^2 \mid x = [c : d : 0]\} = \emptyset.$$

*Proof.* Since  $[1 : 0 : 0]$  and  $[0 : 1 : 0] \notin \overline{\mathcal{X}_n}$  it is sufficient to consider  $x = [1 : s : 0]$  with  $s \neq 0$  and show that  $x \notin \overline{\mathcal{X}_n} \cap \overline{\mathcal{Y}_m}$ . First note that if  $f_{c,d} \in \mathcal{R}$ , then

$$f_{c,d}^2(z) = 1 + \frac{z^2}{\frac{z^2}{c} + z + \frac{d}{c}} + \frac{d}{c^2} \frac{z^4}{\left(\frac{z^2}{c} + z + \frac{d}{c}\right)^2}.$$

Thus as  $[c : d : 1]$  converges to  $x$ ,

$$f_{c,d}^2(z) \rightarrow 1 + \frac{z^2}{z + s} = g_s(z)$$

uniformly in compact subsets of  $\mathbb{C} \setminus \{-s\}$ .

Note that  $g_s$  has a parabolic fixed point at  $\infty$ , and that  $g_s(-s) = \infty$ . Moreover, since one of the forward orbits of the two critical points  $0, -2s$  of  $g_s$  must be infinite and converge to  $\infty$ , one of the two critical points of  $g_s$ , call it  $\omega$ , has an infinite forward orbit entirely contained in  $\mathbb{C} \setminus \{-s\}$ .

If  $[c : d : 1]$  is sufficiently close to  $x$ , then the first  $\max\{n, m\} + 1$  iterates of  $\omega_1 = 0$  (when  $\omega = 0$ ) or of  $\omega_2 = -2d/c$  (when  $\omega = -2s$ ) are pairwise distinct. Thus, for all  $[c : d : 1]$  sufficiently close to  $x$ , we have that  $[c : d : 1] \notin \mathcal{X}_n \cup \mathcal{Y}_m$ . Therefore,  $x \notin \overline{\mathcal{X}_n} \cap \overline{\mathcal{Y}_m}$ .  $\square$

**2.4. Counting intersections at  $d = 0$ .** Now our aim is to count the intersections at  $d = 0$ . We start by establishing where the intersections occur.

**Lemma 2.3.** *For all  $n \geq 3$ , if  $[c : 0 : 1] \in \overline{\mathcal{X}_n}$  then  $c^{-1} = -4 \cos^2 \pi p/q$  for some  $1 \leq p < q \leq n$  with  $(p, q) = 1$  and  $q \neq 2$ .*

*Proof.* For all  $(c, d) \in \mathcal{R}$ , we have that the cross ratio

$$[\omega_1, \omega_2, f_{c,d}(\omega_1), f_{c,d}(\omega_2)] = \frac{c^3}{8d^2} - \frac{c}{2d},$$

where  $\omega_1 = 0, \omega_2 = -2d/c$ . It follows that, given  $c_0 \neq 0$  and a sequence  $(c_k, d_k) \in \mathcal{R} \cap \mathcal{X}_n$  which converges to  $(c_0, 0)$ , the conjugacy class of  $f_{c_k, d_k}$  diverges to infinity in moduli space. According to [7, Lemma 4.1]),  $f_{c_k, d_k}$  has three fixed points, one with multiplier diverging to  $\infty$  and the other two multipliers converge to reciprocal roots of unity of order  $q$  where  $2 \leq q \leq n$ . Uniformly on compact subsets of  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ , the maps  $f_{c_k, d_k}$  converge to

$$M_{c_0}(z) = 1 + \frac{c_0}{z}.$$

Since  $\mathbb{C}^*$  contains the fixed points of  $M_{c_0}$ , the multiplier of the fixed points of  $M_{c_0}$  are of the form  $\exp(\pm 2\pi i p/q)$  where  $(p, q) = 1$ . It follows that

$$c_0^{-1} = -4 \cos^2(\pi p/q).$$

In particular,  $q \neq 2$  and the lemma follows.  $\square$

The rest of this section is devoted to prove the formula below that computes the intersection numbers at the relevant points of  $d = 0$ .

**Proposition 2.4.** *Consider  $n \geq 3$  and  $m \geq n$ . Let  $1 \leq p < q \leq n$  with  $(p, q) = 1$  and  $q \neq 2$ . If  $c_{p,q}^{-1} = -4 \cos^2(\pi p/q)$ , then*

$$\overline{\mathcal{X}_n} \cdot_{c_{p,q}} \overline{\mathcal{Y}_m} = \nu_q(n) \nu_q(m).$$

To prove the above proposition it is convenient to work with the *totally marked moduli space*  $\mathcal{M}_2^{tm}$  which is a smooth two complex dimensional manifold (see [7, Lemma 6.6.]). Namely, the space of quadratic rational maps with marked critical points and marked fixed points. The elements of  $\mathcal{M}_2^{tm}$  are conjugacy classes of sextuples  $(f, \omega_1, \omega_2, x_1, x_2, x_3)$  where  $f$  is a quadratic rational map,  $\omega_j$  are critical points of  $f$ , and  $x_1, x_2, x_3$  is a complete list of the fixed of  $f$  (with repetitions when  $f$  has a multiple fixed point). As before, two such sextuples are conjugate if there is a Möbius transformation conjugating the rational maps and respecting markings. The conjugacy class of  $(f, \omega_1, \omega_2, x_1, x_2, x_3)$  will be denoted by  $[f, \omega_1, \omega_2, x_1, x_2, x_3]$ .

In  $\mathcal{M}_2^{tm}$  we let  $A_n$  (resp.  $B_m$ ) be the curves where the first (resp. the second) critical point is periodic of period  $n$  (resp.  $m$ ). Following 7.4 of [11] to study the



ends of  $A_n$  and  $B_m$  it is convenient to consider the family quadratic rational maps defined for  $(\zeta, \rho) \in \mathbb{C}^* \times \mathbb{C} \setminus \{0, -2\}$ :

$$\begin{aligned} h_{\zeta, \rho}(z) &= \zeta z \left(1 - \frac{2 + \rho}{2(1 + \rho)} z\right) \left(1 - \frac{2}{2 + \rho} z\right)^{-1} \\ &= \zeta z \left(1 - \frac{\rho^2 z}{4(1 + \rho)(1 + \frac{1}{2}\rho - z)}\right) = \zeta z P_\rho(z). \end{aligned}$$

A similar parametrisation was used by James Stimson in his thesis [13].

The critical points of  $h_{\zeta, \rho}$  are  $\omega_1 = 1$  and  $\omega_2 = 1 + \rho$ . There are two distinguished fixed points, one at  $x_1 = 0$  and the other at  $x_2 = \infty$ . Therefore, we may parametrise a subset  $\mathcal{S}$  of  $\mathcal{M}_2^{tm}$  as follows by  $\mathbb{C}^* \times \mathbb{C} \setminus \{0, -2\}$  by  $[h_{\zeta, \rho}, 1, 1 + \rho, 0, \infty, x_3]$  where  $x_3$  is the remaining fixed point.

A partial compactification of  $\mathcal{S}$  is achieved by adding the line  $\rho = 0$ . That is, we identify  $\mathcal{S}$  with a subset of  $S = \mathbb{C}^* \times \mathbb{C} \setminus \{2\}$ . Now let  $\mathcal{A}_n = A_n \cap \mathcal{S}$  and  $\mathcal{B}_m = B_m \cap \mathcal{S}$ . Consider the closures  $\overline{\mathcal{A}_n}$  and  $\overline{\mathcal{B}_m}$  in  $S$ . The following was proven in Stimson's Thesis.

**Theorem 2.5** (Stimson). *The following statements hold:*

- (1) *The intersection of  $\overline{\mathcal{A}_n}$  (resp.  $\overline{\mathcal{B}_n}$ ) with the line  $\rho = 0$  is equal to*

$$\{\exp(2\pi i p/q) \mid 1 \leq p < q \leq n\}.$$

- (2) *Let  $1 \leq p < q \leq n$  such that  $p$  and  $q$  are relatively prime. The intersection number of  $\overline{\mathcal{A}_n}$  (resp.  $\overline{\mathcal{B}_n}$ ) with the line  $\rho = 0$  at  $\kappa = \exp(2\pi i p/q)$  is equal to the number of hyperbolic components of period  $n$  in the  $p/q$ -limb of the Mandelbrot set.*

- (3) *Let  $E$  (resp.  $F$ ) be an irreducible germ of the curve  $\overline{\mathcal{A}_n}$  (resp.  $\overline{\mathcal{B}_n}$ ) at  $\kappa$ . Then,*

$$\frac{\zeta - \kappa}{\kappa \rho} \rightarrow a_1 \in \{a \in \mathbb{C} \mid \Re a > 0\},$$

as  $E \ni (\zeta, \rho) \rightarrow (\kappa, 0)$  and,

$$\frac{\zeta - \kappa}{\kappa \rho} \rightarrow a_2 \in \{a \in \mathbb{C} \mid \Re a < 0\},$$

as  $F \ni (\zeta, \rho) \rightarrow (\kappa, 0)$ .

**Remark** Part (3) of the Theorem above might not be explicitly stated in Stimson's Thesis. However, it can be deduced as follows. According to Stimson, if  $\rho \rightarrow 0$  then  $(\zeta - \kappa)/(\kappa \rho) \rightarrow a$  for some  $a \in \mathbb{C}$ , and, uniformly in compact subsets of  $\mathbb{C} \setminus \{1/2\}$ ,

$$\frac{h^q(1 + z\rho) - 1}{\rho} \rightarrow a + z + \frac{1}{2(2z - 1)} = g_a(z).$$

The map  $g_a$  has a parabolic fixed point at  $\infty$  and critical points at  $z = 0$  and  $z = 1$ . If  $\Re a \leq 0$ , it is not difficult (by a direct calculation) to check that  $\Re g_a^k(0)$  is a strictly decreasing sequence and  $g_a^k(0)$  diverges to  $\infty$ . Similarly, if  $\Re a \geq 0$ , then  $\Re g_a^k(1)$  is a strictly increasing sequence and  $g_a^k(1)$  diverges to  $\infty$ . Part (3) of the above theorem follows.

**Lemma 2.6.** *Let  $1 \leq p < q \leq \min\{n, m\}$  with  $(p, q) = 1$  and  $q \neq 2$ . Let  $c_{p, q} = -4 \cos^{-2}(\pi p/q)$  and  $\kappa = \exp(2\pi i p/q)$ . Then*

$$\overline{\mathcal{V}_n} \cdot_{c_{p, q}} \overline{\mathcal{W}_m} = \overline{\mathcal{A}_n} \cdot_{\kappa} \overline{\mathcal{B}_m}.$$

*Proof.* It is sufficient to show that there is a biholomorphic map between a neighborhood  $U$  of  $c_{p,q}$  in the  $(c, d)$ -plane to a neighborhood  $U'$  of  $\kappa$  in the  $(\zeta, \rho)$ -plane that maps  $\mathcal{V}_n$  to  $\mathcal{A}_n$  and  $\mathcal{W}_m$  to  $\mathcal{B}_m$ . In fact, choosing  $U$  sufficiently small we may mark the analytic continuation of one fixed point of  $M_{c_{p,q}}$  by  $x_1(c, d)$ , the other by  $x_2(c, d)$ . Now for all  $(c, d) \in U$  with  $d \neq 0$ , we have a quadratic rational map  $f_{c,d}$  with marked critical points and marked fixed points. Let  $\zeta(c, d)$  be the multiplier of the fixed point  $x_1$ . Denote by  $[x_1(c, d), 0, x_2(c, d), -2d/c]$  the cross ratio of these four numbers. Let  $\rho(c, d) = [x_1(c, d), 0, x_2(c, d), -2d/c] - 1$ . Note that as  $d \rightarrow 0$ , we have that  $\rho(c, d) \rightarrow 0$ . Thus, we let  $\rho(c, 0) = 0$ . It follows that the element of  $\mathcal{M}_2^{tm}$  determined by  $f_{c,d}$  with these markings is the one given by  $h_{\zeta, \rho}$  for  $\zeta = \zeta(c, d)$  and  $\rho = \rho(c, d)$  whenever  $d \neq 0$ . Therefore the map  $\Phi : U \rightarrow \mathbb{C}^2$  given by  $\Phi(c, d) = (\zeta(c, d), \rho(c, d))$  is holomorphic and its image contains an open neighborhood of  $\kappa$ .

It is easy to see that the map that forgets the marking of the fixed points is also holomorphic and it is the inverse of  $\Phi$ .  $\square$

The proof of proposition 2.4 follows, since (3) of 2.5 implies that at an intersection point of  $\overline{\mathcal{A}_n}$  and  $\overline{\mathcal{B}_m}$  in  $\rho = 0$ , the tangents to  $\overline{\mathcal{A}_n}$  and the tangents to  $\overline{\mathcal{B}_m}$  are distinct.

**2.5. Low periods and the proof of Theorem 1.1.** The last ingredients needed to prove Theorem 1.1 are contained in the lemma below.

**Lemma 2.7.**

$$\eta_{IV}(1, m) = 2^{m-1},$$

$$\eta_{IV}(1, m, p/q) = \nu_q(m),$$

$$\eta_{II}(2) = 1,$$

$$\eta_{IV}(2, m) - \eta_{IV}(1, m) = \frac{1}{3}2^m - \frac{(-1)^m}{3}.$$

Here,  $\eta_{IV}(1, m, p/q)$  denotes the number of type IV hyperbolic components in the  $p/q$ -limb of the Mandelbrot set with an attractive cycle of period dividing  $m$ .

*Proof.* The formula for  $\eta_{IV}(1, m)$  is a consequence of the Douady-Hubbard classification [3] of hyperbolic components in the Mandelbrot set for the parameter family of quadratic polynomials  $\{z^2 + v \mid v \in \mathbb{C}\}$ , which is naturally identified with  $\mathbb{C}$ . The complement of the Mandelbrot set in  $\mathbb{C}$  is the intersection with  $\mathbb{C}$  of the type I hyperbolic component of rational maps which is mentioned in the introduction. The type I hyperbolic component contains no critically finite maps. In fact, it coincides with the set of  $v$  for which the forward orbit of 0 under  $z \mapsto z^2 + v$  diverges to  $\infty$ . The main cardioid of the Mandelbrot set (as it is known) is the hyperbolic component of  $z \mapsto z^2$ . The set of all the other hyperbolic components in the Mandelbrot set, for which the attractive cycle is of period dividing  $m$ , is in two-to-one correspondence with the set of odd-denominator rationals

$$\left\{ \frac{k}{2^m - 1} : 0 < k < 2^m - 1 \right\}.$$

Therefore, we obtain

$$\eta_{IV}(1, m) = 1 + \frac{2^m - 2}{2} = 2^{m-1}.$$

Furthermore, the set of all hyperbolic components in the  $p/q$  limb of the Mandelbrot set, for which the attractive cycle is of period dividing  $m$ , is in two-to-one correspondence with the set of odd-denominator rationals  $k/(2^m - 1)$  in a closed interval  $[1/(2^q - 1), 2/(2^q - 1)]$ , since the arguments of the root are of the form  $r/(2^q - 1)$  and  $(r + 1)/(2^q - 1)$  for some  $r$  coprime to  $q$ . This is the number  $\nu_q(m)$ .

The fact that  $\eta_{II}(2) = 1$  follows immediately from fixing critical points of a quadratic rational map to be 0 and  $\infty$ . If these are in a period two cycle then the map must be of the form  $z \mapsto \lambda z^{-2}$  for some  $\lambda \neq 0$ , and any such maps are conjugate to  $z \mapsto z^{-2}$ , by a conjugacy of the form  $z \mapsto \mu z$ .

We now give a proof of the formula for  $\eta_{IV}(2, m) - \eta_{IV}(1, m)$  which is close to the methods of section 2.4. Since a great deal is known about  $V_2$ , other proofs are possible. For example, one can use the results of a recent paper of Aspenberger-Yampolsky, [1], some of which occur also in work of Timorin [15], and both of which consolidate the structure that has been known in outline for some time, as evidenced, for example, by the thesis of Luo [5] in the 1990's. One can also make use of the theory of matings, which involves using Thurston's criterion [4] and Tan Lei's theorem to decide which matings are realisable [14].

Instead of using  $\mathcal{R}$  as in the case of  $V_n$  for  $n \geq 3$ , we use a parametrisation which extends that used in [1] and [15]:

$$g_{a,b}(z) = \frac{a}{z^2 + 2z + b}.$$

If  $a \neq 0$ , critical points of  $g_{a,b}$  are  $\infty$  and  $-1$ , and  $g_{a,b}(\infty) = 0$ . Every Möbius conjugacy class in  $\mathcal{M}_2^m \setminus X_1$ , apart from that of  $z \mapsto z^{-2}$ , is represented by exactly one map  $g_{a,b}$  for  $a \neq 0$ , and therefore  $\mathcal{M}_2^m \setminus (X_1 \cup \{z^{-2}\})$  can be identified with

$$\{[a : b : 1] \mid a, b \in \mathbb{C}, a \neq 0\} \subset \mathbb{CP}^2.$$

We write  $\mathcal{Z}_m$  for the set of  $[a : b : 1]$  such that  $g_{a,b}$  represents an element of  $Y_m$ . Then

$$\eta_{IV,2,m} - \eta_{IV,1,m} = \#(\mathcal{Z}_m \cap \{[a : 0 : 1] \mid a \neq 0\}).$$

As in the earlier cases, all intersections are transverse. We write  $\overline{\mathcal{Z}_m}$  for the closure of  $\mathcal{Z}_m$  in  $\mathbb{CP}^2$ . The plane  $b = 0$  in  $\mathbb{CP}^2$  is the set of all  $[a : 0 : 1]$  together with  $[1 : 0 : 0]$ . Provided that all zeros are simple, by Bezout's Theorem,

$$\#(\overline{\mathcal{Z}_m} \cap (b = 0)) = \deg(\overline{\mathcal{Z}_m}).$$

Therefore,

$$\eta_{IV,2,m} - \eta_{IV,1,m} = \deg(\overline{\mathcal{Z}_m}) - \overline{\mathcal{Z}_m} \cdot_{[0:0:1]} (b = 0) - \overline{\mathcal{Z}_m} \cdot_{[1:0:0]} (b = 0)$$

where  $\overline{\mathcal{Z}_m} \cdot_{[0:0:1]} (b = 0)$  denotes the intersection number of  $\overline{\mathcal{Z}_m}$  with  $(b = 0)$  at  $[0 : 0 : 1]$ . To compute  $\deg(\overline{\mathcal{Z}_m})$ , we write

$$g_{a,b}^m(1) = \frac{T_m(a, b)}{U_m(a, b)}$$

Then  $\mathcal{Z}_m$  is the zero set of  $T_m^h + U_m^h$ , where, following the notation of 2.2,  $T_m^h$  and  $U_m^h$  are the homogenized versions of the polynomials  $T_m$  and  $U_m$ . Induction shows

that the degrees of  $T_m$ ,  $U_m$  and  $T_m^h + U_m^h$  are  $2^m - 1$  for all  $m \geq 1$ . As in 2.2 we see that, for all  $m \geq 3$ , there is a birational map  $\psi : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$  such that

$$P_m^h \circ \psi = \begin{cases} T_m^h + U_m^h & \text{if } m \text{ is odd} \\ \frac{T_m^h + U_m^h}{T_2^h + U_2^h} & \text{if } m \text{ is even.} \end{cases}$$

Hence, as in 2.2, since  $P_m^h$  is square-free for all  $m \geq 3$ , and since  $T_m^h + U_m^h$  is square-free  $m = 1$  or  $m = 2$ , this is also true for  $T_m^h + U_m^h$  for all  $m \geq 3$ . So

$$\deg(\mathcal{Z}_m) = \deg(T_m^h + U_m^h) = 2^m - 1.$$

So now we need to compute  $\overline{\mathcal{Z}_m} \cdot_{[0:0:1]} (b = 0)$  and  $\overline{\mathcal{Z}_m} \cdot_{[1:0:0]} (b = 0)$ . We use homogenized coordinates  $[a : b : t]$ . We write

$$T_m(a, b) = T_m^1(a) + bT_m^2(a, b), \quad T_m^h(a, b, t) = T_m^{h,1}(a, t) + bT_m^{h,2}(a, t),$$

$$U_m(a, b) = U_m^1(a) + bU_m^2(a, b), \quad U_m^h(a, b, t) = U_m^{h,1}(a, t) + bU_m^{h,2}(a, t)$$

Then  $\overline{\mathcal{Z}_m} \cdot_{[0:0:1]} (b = 0)$  and  $\overline{\mathcal{Z}_m} \cdot_{[1:0:0]} (b = 0)$  are the maximum powers of  $a$  and  $t$  respectively which divide  $T_m^{h,1} + U_m^{h,1}$ . The first of these is the maximum power of  $a$  dividing  $T_m^1 + U_m^1$ . The second is  $\deg(T_m + U_m) - \deg(T_m^1 + U_m^1)$ . So if we write

$$g_{a,0}^m(-1) = \frac{T_m^0(a)}{U_m^0(a)}$$

where  $T_m^0$  and  $U_m^0$  have no non-zero power of  $a$  common factor, we see that

$$\eta_{IV,2,m} - \eta_{IV,1,m} = \deg(T_m^0 + U_m^0).$$

A straightforward induction gives

$$T_1^0 = a, \quad U_1^0 = -1$$

$$T_2^0 = 1, \quad U_2^0 = a - 2,$$

$$T_{2k+1}^0 = a(U_{2k}^0)^2,$$

$$U_{2k+1}^0 = T_{2k}^0(T_{2k}^0 + 2U_{2k}^0),$$

$$T_{2k+2}^0 = (U_{2k+1}^0)^2 = (T_{2k}^0)^2(T_{2k}^0 + 2U_{2k}^0)^2,$$

$$U_{2k+2}^0 = T_{2k+1}^0(T_{2k+1}^0 + 2U_{2k+1}^0)/a = (U_{2k}^0)^2(a(U_{2k}^0)^2 + 2(T_{2k}^0)^2 + 4T_{2k}^0U_{2k}^0).$$

Induction also gives  $\deg(U_{2k}^0) > \deg(T_{2k}^0)$  for all  $k \geq 1$  and  $\deg(T_{2k+1}^0) > \deg(U_{2k+1}^0)$  for all  $k \geq 0$ . More precisely, we have

$$\deg(T_{2k+1}^0) = 2\deg(U_{2k}^0) + 1,$$

$$\deg(U_{2k+1}^0) = \deg(T_{2k}^0) + \deg(U_{2k}^0),$$

$$\deg(T_{2k+2}^0) = 2\deg(U_{2k+1}^0) = 2(\deg(T_{2k}^0) + \deg(U_{2k}^0)),$$

$$\deg(U_{2k+2}^0) = 4\deg(U_{2k}^0) + 1.$$

This gives

$$\deg(U_{2k}^0) = \sum_{i=0}^{k-1} 4^i = \frac{4^k - 1}{3} = \frac{2^{2k} - 1}{3},$$

$$\deg(T_{2k+1}^0) = \frac{2^{2k+1} + 1}{3}.$$

Since all coefficients of all terms in  $U_m^0$  and  $V_m^0$  are positive for all  $m$ , the degree of  $T_m^0 + U_m^0$  is the maximum of the degrees of  $T_m^0$  and  $U_m^0$ , and we obtain

$$\deg(T_m^0 + U_m^0) = \frac{2^m - (-1)^m}{3},$$

as required. □

### 3. TYPE II COMPONENTS

As before, we let  $\eta_{II}(m)$  denote the number of type II components of period dividing  $m$ , in the space  $\mathcal{M}_2^{cm}$ , and  $\eta'_{II}(m)$  is the number of type II components of period exactly  $m$ . For  $1 \leq j < m$ , let  $\eta_{II}(m, j)$  denote the number of type II components of period  $\geq 3$  dividing  $m$  such that the second marked critical point  $\omega_2$  is in the  $j$ 'th iterate of the immediate attractive basin of  $\omega_1$ . Note that

$$\sum_{d|m, d \geq 3} \frac{m}{d} \eta'_{II}(d) = \sum_{j=1}^{m-1} \eta_{II}(m, j).$$

We have already seen that  $\eta_{II}(2) = \eta_{II}(2, 1) = 1$ . The aim of this section is to prove Theorem 1.2. In fact, we obtain a sharper result by giving a formula for  $\eta_{II}(m, j)$ . Here  $\nu'_q(j)$  is as in Section 1.

**Theorem 3.1.** *For all  $m \geq 3$ , and all  $1 \leq j < m$ ,*

$$\begin{aligned} \eta_{II}(m, j) = & \frac{7}{36} 2^m - \frac{1}{12} (2^j + 2^{m-j}) - \frac{(-1)^m}{36} ((-2)^j + (-2)^{m-j}) \\ (3.1) \quad & - \frac{1}{4} - \frac{5}{36} (-1)^m + \frac{1}{12} ((-1)^j + (-1)^{m-j}) \\ & - \frac{1}{2} \sum_{3 \leq q} \phi(q) \nu'_q(j) \nu'_q(m-j). \end{aligned}$$

Hence

$$\begin{aligned} (3.2) \quad \sum_{d|m, d \geq 3} \frac{m}{d} \eta'_{II}(d) = & \frac{7}{36} m 2^m - \frac{37}{108} 2^m - \frac{m}{4} - (-1)^m \frac{5}{36} m + \frac{1}{2} + (-1)^m \frac{5}{54} \\ & - \frac{1}{2} \sum_{3 \leq q, q < j < m-q} \phi(q) \nu'_q(j) \nu'_q(m-j). \end{aligned}$$

**Remark 1.** Note that the formula should be symmetric in  $j$  and  $m-j$ , because we can interchange  $\omega_1$  and  $\omega_2$ .

2. In particular,

$$\eta_{II}(3, 1) = \eta_{II}(3, 2) = 1,$$

$$\eta_{II}(4, 1) = \eta_{II}(4, 2) = \eta_{II}(4, 3) = 2,$$

$$\eta_{II}(5, j) = 5 \quad \text{for } 1 \leq j \leq 4,$$

$$\eta_{II}(6, j) = \begin{cases} 10 & \text{for } j = 1, 2, 4, 5, \\ 11 & \text{for } j = 3, \end{cases}$$

$$\eta_{II}(7, j) = \begin{cases} 21 & \text{for } j = 1, 2, 5, 6 \\ 23 & \text{for } j = 3, 4, \end{cases}$$

$$\eta_{II}(8, j) = \begin{cases} 42 & \text{for } j = 1, 2, 6, 7 \\ 47 & \text{for } j = 3, 5, \\ 45 & \text{for } j = 4. \end{cases}$$

The only non-zero contribution to  $\eta_{II}(m, j)$  from the last row of (3.1), in this list, is when  $m = 8$ ,  $j = 4$  and  $q = 3$ , when  $\nu'_3(4) = 1$ .

**3.1. Outline Proof.** We use the representation of the elements of  $\mathcal{R} = \mathcal{M}_2^{cm} \setminus X_2$  by maps  $f_{c,d}$  where  $(c, d) \in \mathbb{C} \times \mathbb{C}^*$  as in Section 2.1. Given  $m \geq 3$ , in  $\mathcal{R}$ , for  $j \geq 1$  we consider the curves:

$$\mathcal{P}_j = \{(c, d) \in \mathcal{R} \mid f_{c,d}^j(0) = -\frac{2d}{c}\}$$

and

$$\mathcal{Q}_{m-j} = \{(c, d) \in \mathcal{R} \mid f_{c,d}^{m-j}\left(-\frac{2d}{c}\right) = 0\}.$$

The number  $\eta_{II}(m, j)$  is then given by the cardinality of  $\mathcal{P}_j \cap \mathcal{Q}_{m-j}$ . All intersections between  $\mathcal{P}_j$  and  $\mathcal{Q}_{m-j}$  are transverse, because the subset of a type II hyperbolic component in which one critical point maps onto the other consists of two transverse submanifolds. This is proved in, for example, [9], where parametrisations are given for the hyperbolic components of each type.

As for type IV components we consider the closure of  $\mathcal{P}_j$  and  $\mathcal{Q}_{m-j}$  in  $\mathbb{CP}^2$  and apply Bezout's Theorem.

We will start by computing the degree of  $\overline{\mathcal{P}_j}$  and  $\overline{\mathcal{Q}_{m-j}}$  and then continue to subtract the intersections at "infinity" from the product of the degrees. The extra difficulty for counting intersections at infinity arises from the fact that we will have to establish an analogue of part (2) of Stimson's Theorem 2.5. Both, computing the degrees and computing the intersections at infinity will heavily rely on parametrisation of the unique Type I component introduced in [9].

**3.2. The  $v$  coordinate.** We proceed to summarize the relevant results contained in [9] related to the parametrisation of the Type I component.

For each  $\zeta \in \mathbb{D}^* = \{\zeta \mid 0 < |\zeta| < 1\}$  we consider the quadratic rational map

$$\tau_\zeta(z) = z \frac{z + \zeta}{1 + \bar{\zeta}z}.$$

Note that both  $z = 0$  and  $z = \infty$  are attracting fixed points of  $\tau_\zeta$ . The basin of  $z = 0$  is  $\mathbb{D}$  and contains a unique critical point which we denote by  $c(\zeta)$ . The dynamics in  $\mathbb{D}$  is semiconjugate, via the Königs coordinates, to multiplication by  $\zeta$ . More precisely, there exists a unique holomorphic map  $\phi_\zeta : \mathbb{D} \rightarrow \mathbb{C}$  such that  $\zeta\phi_\zeta(z) = \phi_\zeta \circ \tau_\zeta(z)$  and  $\phi_\zeta(0) = 0, \phi_\zeta(c(\zeta)) = 1$ . Such a map is an isomorphism between a neighborhood of the origin and the unit disk. More precisely, there exists a conformal map  $\sigma_\zeta : \mathbb{D} \rightarrow \sigma_\zeta(\mathbb{D})$  such that  $\sigma_\zeta(0) = 0$  which is an inverse of  $\phi_\zeta$  (that is,  $\phi_\zeta \circ \sigma_\zeta(z) = z$ , and for all  $w \in \sigma_\zeta(\mathbb{D})$ , we also have that  $\sigma_\zeta \circ \phi_\zeta(w) = w$ ).

Now consider an element  $[f, \omega_1, \omega_2] \in \mathcal{M}_2^m$  which lies in the Type I component. Then  $f$  has an attracting fixed point  $z_0$  which we assume it has multiplier  $\zeta \neq 0$ . As above, there exists a conformal isomorphism  $\sigma_f : \mathbb{D} \rightarrow \sigma_f(\mathbb{D})$  with image contained in the basin of  $z_0$  such that  $\sigma_f(0) = z_0, \sigma_f(\zeta z) = f(\sigma_f(z))$  and at least one critical point of  $f$  lies in  $\partial\sigma_f(\mathbb{D})$ . (This map  $\sigma_f$  always extends to a homeomorphism from  $\overline{\mathbb{D}}$  onto its image.)

Let  $\mathcal{U}$  be the subset of the Type I component formed by all  $[f, \omega_1, \omega_2]$  with an attracting fixed point of non vanishing multiplier such that  $\omega_1 \notin \partial\sigma_f(\mathbb{D}) \ni \omega_1$ . Note that  $\mathcal{U}$  is well defined since the required properties are invariant under conjugacies that respect critical markings.

Given  $[f, \omega_1, \omega_2] \in \mathcal{U}$  with attracting fixed point  $z_0$  with multiplier  $\zeta = \zeta(f)$  and basin of attraction  $U_f$ , there exists a unique Königs coordinate  $\phi_f : U \rightarrow \mathbb{C}$  that semiconjugates  $f$  with multiplication by  $\zeta(f)$  such that  $\phi_f(\omega_2) = 1$ . In the sequel we will always assume that  $\phi_f$  is this unique semiconjugacy. In a neighborhood of  $z_0$  the map  $\sigma_\zeta^{-1} \circ \phi_f$  is a conjugacy between  $f$  and  $\tau_\zeta$ . Taking iterated preimages, the conjugacy  $\sigma_\zeta^{-1} \circ \phi_f$  uniquely extends to a simply connected domain contained in  $U_f$  containing the critical value  $f(\omega_1)$ . The  $v$ -coordinate of  $f \in \mathcal{U}$  is, by definition,  $v(f) = \sigma_\zeta^{-1} \circ \phi_f(f(\omega_1))$ .

According to [9],

$$\begin{aligned} \mathcal{U} &\rightarrow \mathbb{D}^* \times \mathbb{D} \\ [f, \omega_1, \omega_2] &\mapsto (\zeta(f), v(f)) \end{aligned}$$

is a real-analytic homeomorphism.

### 3.3. The degrees of the curves.

**Lemma 3.2.** *Let  $m \geq 3$ . The following statements hold:*

- For all  $j \geq 1$ , the degree of  $\overline{\mathcal{P}_j}$  is

$$\frac{1}{6}(2^j - (-1)^j) + \frac{1}{2}.$$

- For all  $j \geq 1$ , the degree of  $\overline{\mathcal{Q}_{m-j}}$  is

$$\frac{1}{6}(7 \cdot 2^{m-j} - (-1)^{m-j}) - \frac{1}{2}.$$

*Proof.* We write  $(cP_j + 2dQ_j)^h$  for the homogenization of  $cP_j + 2dQ_j$ . Similarly,  $R_{m-j}^h$  is the homogenization of  $R_{m-j}$  where  $P_j, Q_j, R_{m-j}$  are as in Section 2.2. Observe that  $\overline{\mathcal{P}_j}$  and  $\overline{\mathcal{Q}_{m-j}}$  are the varieties of  $(cP_j + 2dQ_j)^h$  and  $R_{m-j}^h$ , respectively. Using the calculation of 2.2, we have

$$\begin{aligned} \deg((cP_j + 2dQ_j)^h) &= \frac{1}{6}(2^j - (-1)^j) + \frac{1}{2}, \\ \deg(R_{m-j}^h) &= \frac{7}{6}2^{m-j} - \frac{1}{6}(-1)^{m-j} - \frac{1}{2}. \end{aligned}$$

Hence to establish the lemma we need to show that these polynomials are square free. From the birational equivalence argument we just need to check that  $cP_j + 2dQ_j$  is square-free.

Given  $0 < r < 1$ , let  $L_r$  be the set formed by the maps  $f_{c,d}$  in  $\mathcal{R}$  that have an attracting fixed point with multiplier  $r$ . From [9], the intersection of  $\mathcal{P}_j$  with  $L_r$  is contained in  $\mathcal{U}$  and, as summarized above, it is in bijection with points  $z \in \mathbb{D}$  which map onto  $c(r)$  under  $j-1$  iterates of map  $\tau_r$ . That is, the cardinality of the intersection  $\mathcal{P}_j \cap L_r$  is  $2^{j-1}$ , all of them transverse.

It is sufficient to show that the number of intersections of  $\overline{\mathcal{P}_j}$  and  $\overline{L_r}$  coincides with the product of the degrees of  $cP_j + 2dQ_j$  and the polynomial equation of  $L_r$  which we proceed to compute.

The curve  $L_r$  is given by the equations

$$1 + \frac{c}{z} + \frac{d}{z^2} = z, \quad -\frac{c}{z^2} - 2\frac{d}{z^3} = r.$$

These equations give

$$z = \frac{1 \pm \sqrt{1 + c(2+r)}}{2+r}$$

Substituting for  $z$ , the curve  $\zeta = r$  becomes the degree three curve in  $c$  and  $d$ :

$$\begin{aligned} & \frac{c^2 r^2}{(2+r)^4} (8 - c(2+r)) - \frac{c}{(2+r)^4} (c(2+r)^2 + 4r)^2 \\ & + \frac{cr}{(2+r)^5} (c(2+r)^2 + 4r)(4 - 2c(2+r)) + 4d^2 + \frac{4d}{(2+r)^3} (c(2+r)^2 + 4r) \\ & + \frac{12dcr}{(2+r)^2} = 0 \end{aligned}$$

The only term of degree three is  $c^3$  and so there are no elements of  $\overline{L_r}$  at  $[c : 0 : 1]$  for  $c \neq 0$ . Since both, the above equation of  $L_r$  and  $cP_j + 2dQ_j$  have linear terms but no constant terms,  $[0 : 0 : 1] \in \overline{\mathcal{P}_j} \cap \overline{L_r}$  and the intersection number is 1.

Finally,  $[0 : 1 : 0]$  lies in the intersection only when  $j$  is odd. In fact, when  $j$  is even, both  $P_j$  and  $Q_j$  have the same degree and both have powers of  $d$  of maximal degree, thus  $[0 : 1 : 0] \notin \overline{\mathcal{P}_j}$  if  $j$  is even. When  $j$  is odd,  $\deg(P_j) = \deg(Q_j) + 1$ , so  $[0 : 1 : 0] \in \overline{\mathcal{P}_j}$  and the intersection number at  $[0 : 1 : 0]$  is also 1.

Thus, the total intersection number between  $\overline{\mathcal{P}_j}$  and  $\overline{L_r}$  is  $2^{j-1} + 1 + (1 - (-1)^j)/2$  which is equal to  $3 \deg(P_j + 2dQ_j)$ . Hence,  $P_j + 2dQ_j$  is square free.  $\square$

### 3.4. No intersections at $c = d = \infty$ and intersections at $c = 0$ .

**Lemma 3.3.** *We have the following.*

- $[1 : s : 0] \notin \overline{\mathcal{P}_j} \cap \overline{\mathcal{Q}_{m-j}}$  for all  $s \in \mathbb{C}$ .
- $\overline{\mathcal{P}_j} \cdot_{[0:0:1]} \{[c : 0 : 1]\} = 1$  and  $\overline{\mathcal{P}_j} \cdot_{[0:0:1]} \overline{\mathcal{Q}_{m-j}} = 2^{m-j-1}$  for all  $1 \leq j < m$ ,
- $\overline{\mathcal{P}_j} \cdot_{[0:1:0]} \overline{\mathcal{Q}_{m-j}} = \frac{1}{6} (1 - (-1)^j) (2^{m-j} - (-1)^{m-j})$ .

*Proof.* *Intersections at  $[1 : s : 0]$ .* For  $s \neq 0$  this follows directly from Lemma 2.2, since  $\overline{\mathcal{P}_j} \cap \overline{\mathcal{Q}_{m-j}} \subset \overline{\mathcal{X}_m} \cap \overline{\mathcal{Y}_m}$ . There are no intersections at  $[1 : 0 : 0]$  either, because  $cP_j + 2dQ_j$  always has a term  $c^k$  of maximal degree, since all coefficients in  $P_j$  and  $Q_j$  are positive and  $cP_j$  has such a term.

*Intersections at  $[0 : 0 : 1]$ .* By induction, using the recursive definition of  $R_{k+1}$  and  $S_{k+1}$  in terms of  $R_k$  and  $S_k$  in 2.2, it is not difficult to establish that the minimum



degree  $\min \deg(R_{m-j})$  of the monomials in  $R_{m-j}(c, d)$  is  $2^{m-j-1}$  and that this is uniquely realised by a monomial in  $d$  for all  $1 \leq j < m$ . We saw, in Section 2, that both  $P_j$  and  $Q_j$  have non-zero constant terms for all  $j \geq 3$ . Hence  $cP_j + 2dQ_j$  has non-zero linear terms in  $c$  and  $d$  and the claim follows for all  $j \geq 3$ . Since  $\mathcal{P}_1 = \{c = 0\}$  and  $\mathcal{P}_2 = \{c + 2d = 0\}$ , the claim also follows for  $j = 1, 2$ .

*Intersections at  $[0 : 1 : 0]$ .* There are no intersections if  $j$  is even, because then, as we saw in 2.2, both  $P_j$  and  $Q_j$  have the same degree, and both have powers of  $d$  of maximal degree. So the degree of  $cP_j + 2dQ_j$  is realised by a power of  $d$  in  $2dQ_j$ . But  $\deg(P_j) = \deg(Q_j) + 1$  for all odd  $j \geq 3$ , and hence the degree of intersection of  $(cP_j + 2dQ_j)^h = 0$  with  $c = 0$  is 1 for all odd  $j \geq 3$  (using, again, that linear terms in  $c$  and  $d$  are non-zero). The degree of the intersection of  $(cP_j + 2dQ_j)^h$  with  $R_{m-j}^h$  is then given by  $\deg(R_{m-j}) - \deg(R_{m-j}^1)$ , where we write

$$R_k(c, d) = R_k^1(d) + cR_k^2(c, d), \quad S_k(c, d) = S_k^1(d) + cS_k^2(c, d).$$

Inductively we see that

$$R_1^1 = S_1^1 = 4d, \\ R_{k+1}^1 = (R_k^1)^2 + d(S_k^1)^2, \quad S_{k+1}^1 = (R_k^1)^2,$$

and hence for all  $k \geq 1$ ,

$$\deg(R_{2k+1}^1) = 2 \deg(R_{2k}^1) = 1 + 4 \deg(R_{2k-1}^1).$$

It follows that

$$\deg(R_{m-j}^1) = \frac{5}{6}2^{m-j} - \frac{1}{2} + \frac{1}{6}(-1)^{m-j},$$

and hence

$$\deg(R_{m-j}) - \deg(R_{m-j}^1) = \frac{1}{3}(2^{m-j} - (-1)^{m-j}).$$

□

**3.5. Preliminaries on intersections at  $[c : 0 : 1]$ .** The intersections between  $\overline{\mathcal{P}}_j$  and  $\overline{\mathcal{Q}}_{m-j}$  at  $[c : 0 : 1]$  are also at the points  $c_{p,q}$ .

**Lemma 3.4.** *If  $[c : 0 : 1] \in \overline{\mathcal{P}}_j \cup \overline{\mathcal{Q}}_j$ , then  $c = 0$  or,  $q \geq 3$  and  $c = c_{p,q}$  for some  $p$  with  $(p, q) = 1$  and  $c_{p,q}$  as in Proposition 2.4, that is,  $c_{p,q}^{-1} = -4 \cos^2(\pi p/q)$ .*

*Proof.* If  $f_{c,0} \in \overline{\mathcal{P}}_j$  for  $c \neq 0$ , then it is the limit of maps  $f_{c_n, d_n}$  in  $\mathcal{P}_j$  for a sequence  $(c_n, d_n)$ . The critical points of  $f_{c_n, d_n}$  are 0 and  $-2d_n/c_n$ , where  $-2d_n/c_n \rightarrow 0$ . The critical values are  $\infty$  and  $1 - c_n/4d_n^2$  which converges to  $\infty$ . and hence  $f_{c_n, d_n}^2(0)$  converges to 1. Since  $f_{c_n, d_n}^j(0) = -2d_n/c_n$  and since the maps  $f_{c_n, d_n}$  converge uniformly to  $f_{c,0}$  outside any neighbourhood of 0, restricting to a subsequence there must be some least integer  $3 \leq k \leq j$  such that  $f_{c_n, d_n}^k(0) \rightarrow 0$ , and hence  $f_{c,0}^k(0) = 0$  and  $f_{c,0}$  has order  $k$ . So  $c = c_{p,q}$  as claimed. The proof for  $\overline{\mathcal{Q}}_j$  is similar. □

In order to compute intersection numbers, we use the family of maps  $h_{\zeta, \rho}$  introduced in Proposition 2.4, and varieties  $\mathcal{C}_j$  and  $\mathcal{D}_{m-j}$  corresponding to Möbius conjugates in the  $h_{\zeta, \rho}$  family of the maps  $f_{c, d}$  in  $\mathcal{P}_j$  and  $\mathcal{Q}_{m-j}$ . That is,  $\mathcal{C}_j$  consists of those parameters  $(\zeta, \rho)$  such that the critical point  $\omega_1 = 1$  of  $h_{\zeta, \rho}$  maps in  $j$  iterates onto the critical point  $\omega_2 = 1 + \rho$ . Similarly,  $\mathcal{D}_{m-j}$  consists of those parameters  $(\zeta, \rho)$  such that the critical point  $\omega_2 = 1 + \rho$  of  $h_{\zeta, \rho}$  maps in  $j$  iterates onto the critical point  $\omega_1 = 1$ . As in Theorem 2.5 we may restrict our attention to

study the intersections of  $\overline{\mathcal{C}}_j$  and  $\overline{\mathcal{D}}_{m-j}$  with the line  $\rho = 0$ . In particular, we will only be interested on parameters  $(\zeta, \rho)$  where  $|\rho|$  is close to 0.

**Lemma 3.5.** *Let  $\overline{\mathcal{C}}_j$  and  $\overline{\mathcal{D}}_{m-j}$  be the varieties in the  $(\zeta, \rho)$  coordinates that correspond to  $\overline{\mathcal{P}}_j$  and  $\overline{\mathcal{Q}}_{m-j}$ . Then, for  $c_{p,q}^{-1} = -4\cos^2(\pi p/q)$  and  $\kappa = e^{2\pi ip/q}$ ,*

(1) *The following numbers coincide:*

$$\overline{\mathcal{P}}_j \cdot_{[c_{p,q}:0:1]} \{[c:0:1] \mid c \in \mathbb{C}\}$$

$$\overline{\mathcal{Q}}_j \cdot_{[c_{p,q}:0:1]} \{[c:0:1] \mid c \in \mathbb{C}\}$$

$$\overline{\mathcal{C}}_j \cdot_{\kappa} \{(\zeta, 0) \mid \zeta \in \mathbb{C}\}$$

$$\overline{\mathcal{D}}_j \cdot_{\kappa} \{(\zeta, 0) \mid \zeta \in \mathbb{C}\}.$$

(2)

$$\overline{\mathcal{P}}_j \cdot_{[c_{p,q}:0:1]} \overline{\mathcal{Q}}_{m-j} = (\overline{\mathcal{P}}_j \cdot_{[c_{p,q}:0:1]} \{[c:0:1] \mid c \in \mathbb{C}\}) \cdot (\overline{\mathcal{Q}}_{m-j} \cdot_{[c_{p,q}:0:1]} \{[c:0:1] \mid c \in \mathbb{C}\}),$$

*Proof.* For (1) note that the biholomorphism of the proof of Lemma 2.6 establishes that the first and third number coincide, as well as the second and fourth. Now  $(\zeta, \rho) \mapsto (\zeta, -\rho/(1+\rho))$  interchanges  $\mathcal{C}_j$  and  $\mathcal{D}_j$ . Hence, their intersection numbers at  $\kappa$  coincide.

(2) Once we have established (1), this is proved very similarly to the absence of common tangent lines at  $\rho = 0$  of  $\mathcal{A}_m$  and  $\mathcal{B}_n$  in Theorem 2.5. As there, we have

$$\frac{h^q(1+z\rho) - 1}{\rho} \rightarrow a + z + \frac{1}{2(2z-1)} = g_a(z).$$

If  $\Re a \leq 0$ , then  $\Re g_a^k(0)$  is a strictly decreasing sequence and if  $\Re a \geq 0$ , then  $\Re g_a^k(1)$  is a strictly increasing sequence. Therefore we must have  $\zeta = \kappa(1 + a_1\rho + o(\rho))$  with  $\Re(a_1) > 0$  on any branch of  $\mathcal{C}_j$  near  $(\kappa, 0)$  and  $\zeta = \kappa(1 + a_2\rho + o(\rho))$  with  $\Re(a_2) < 0$  on any branch of  $\mathcal{D}_{m-j}$  near  $(\kappa, 0)$ . The absence of common tangent lines statement follow and (2) is a consequence of this and the biholomorphism of the proof of Lemma 2.6.  $\square$

### 3.6. Intersection number at $\kappa$ .

**Theorem 3.6.** *Let  $\kappa = \exp(2\pi ip/q)$ . then*

$$\overline{\mathcal{C}}_j \cdot_{\kappa} \{(\zeta, 0) \mid \zeta \in \mathbb{C}\} = \nu'_q(j).$$

To compute  $\mathcal{C}_j \cdot_{\kappa} \{(\zeta, 0) \mid \zeta \in \mathbb{C}\}$  we observe that, there exists  $\delta > 0$ , such that this number coincides with the cardinality of

$$\mathcal{C}_j \cap (\{r\kappa\} \times \{\rho \mid |\rho| \leq \delta\})$$

for all  $0 < r < 1$  sufficiently close to 1. According to [9], the parameters in the above intersection correspond to maps in the subset  $\mathcal{U}$  of the Type I component introduced in Section 3.2. Our aim now is to further understand the image of the above intersection under the parametrisation described in Section 3.2.

In order to be more precise, we will state below the relevant results in a lemma and a proposition, and postpone their proofs. Then, using some properties of the numbers  $\nu'_q(j)$  discussed in the Appendix we will prove the theorem. Then we will proceed with the corresponding proofs of the lemma and the proposition.

Recall that  $\phi_{\zeta} : \mathbb{D} \rightarrow \mathbb{C}$  denotes the (normalized) Königs coordinate for  $\tau_{\zeta}$ .

**Lemma 3.7.** *Consider the graph*

$$S_{\kappa} = \{t\kappa^i \mid 1 \leq i \leq q, t \geq 0\} \subset \mathbb{C}$$

Given  $0 < r < 1$ , let  $\zeta = r\kappa$  and consider

$$\Gamma = \Gamma_\zeta = \phi_\zeta^{-1}(S_\kappa) \subset \mathbb{D}.$$

Then the following statements hold:

- $\Gamma$  is connected, simply connected, and locally homeomorphic to a finite tree.
- $\Gamma \setminus \{0\}$  has exactly  $q$  connected components.
- Label by  $\gamma_1^\zeta$  the connected component of  $\Gamma \setminus \{0\}$  containing  $\tau_\zeta(c(\zeta))$ . Then

$$\#\{z \in \gamma_1^\zeta \mid \tau_\zeta^{j-1}(z) = c(\zeta)\} = \nu'_q(j).$$

**Proposition 3.8.** *Let  $X$  be a branch of  $\bar{\mathcal{C}}_j$  at  $(\kappa, 0)$ . There exist  $0 < r_0 < 1$  and  $\delta > 0$  such that if  $h = h_{r\kappa, p} \in X$  for some  $r_0 < r < 1$  and  $|\rho| < \delta$ , then  $v(h) \in \gamma_1^{r\kappa}$ .*

*Proof of Theorem 3.6.* From the previous proposition and lemma we obtain that

$$\bar{\mathcal{C}}_j \cdot_\kappa \{(\zeta, 0) \mid \zeta \in \mathbb{C}\} \leq \nu'_q(j).$$

Applying Bezout's Theorem we obtain the first line below. From Lemmas 3.4 we obtain the second identity. The third line is obtained putting together the inequality above with Lemma 3.5 and Lemma 3.3. Then the fourth line is a consequence of Proposition 4.2 while the last line is Lemma 3.2.

$$\begin{aligned} \deg \bar{\mathcal{P}}_j &= \bar{\mathcal{P}}_j \cdot \{[c : 0 : 1] \mid c \in \mathbb{C}\} \\ &= \bar{\mathcal{P}}_j \cdot_{[0:0:1]} \{[c : 0 : 1] \mid c \in \mathbb{C}\} + \sum_{c_{p,q}, 3 \leq q < j} \bar{\mathcal{P}}_j \cdot_{c_{p,q}} \{[c : 0 : 1] \mid c \in \mathbb{C}\} \\ &\leq 1 + \frac{1}{2} \sum_{3 \leq q < j} \phi(q) \nu'_q(j) \\ &= 1 + \frac{1}{6} (2^j + (-1)^j) - \frac{1}{2} \\ &= \deg \bar{\mathcal{P}}_j. \end{aligned}$$

Thus, equality holds throughout. In particular,

$$\bar{\mathcal{C}}_j \cdot_\kappa \{(\zeta, 0) \mid \zeta \in \mathbb{C}\} = \nu'_q(j).$$

□

**3.6.1. Proof of Lemma 3.7.** We can write  $\Gamma$  as an increasing union of sets  $\Gamma_n$  for  $n \geq 0$ , where

$$\Gamma_n = \tau_\zeta^{-n}(\sigma_\zeta(S_\kappa)).$$

For each  $n$ ,  $\tau_\zeta : \Gamma_{n+1} \rightarrow \Gamma_n$  is a degree two branched cover with a single critical point. So, by induction on  $n$ , each  $\Gamma_n$  is a finite tree. The only intersections between  $\Gamma_{n+1} \setminus \Gamma_n$  and  $\Gamma_n$  are at extreme points of  $\Gamma_n$ , again by induction. It follows that  $\Gamma$  is a connected and locally finite tree. Also by induction,  $\Gamma_n \setminus \{0\}$  has  $q$  components, and each one of the  $q$  components of  $\Gamma_{n+1} \setminus \{0\}$  contains one of the  $q$  components of  $\Gamma_n \setminus \{0\}$ , which, in turn, contains one of the points  $\tau_\zeta^j(0)$  for  $0 \leq j \leq q$ . So taking the union of all of these,  $\Gamma \setminus \{0\}$  also has  $q$  components  $\gamma_j^\zeta$ , for  $1 \leq j \leq q$ , where  $\gamma_j^\zeta$  contains  $\tau_\zeta^j(c(\zeta))$  — and  $\gamma_q^\zeta$  also contains  $c(\zeta)$ . Also,  $\tau_\zeta$  maps  $\gamma_j^\zeta$  homeomorphically onto  $\gamma_{j+1}^\zeta$  if  $1 \leq j \leq q-1$ , and maps  $\gamma_q^\zeta$  onto  $\Gamma$ ,

mapping one-to-one onto  $(\Gamma \setminus \gamma_1^\zeta) \cup \{\tau_\zeta(c(\zeta))\}$  and two-to-one onto  $\gamma_1^\zeta \setminus \{\tau_\zeta(c(\zeta))\}$ . It follows that the number  $a_j(n) = \#(\tau_\zeta^{-n}(c(\zeta)) \cap \gamma_j)$  satisfies

$$a_j(0) = \begin{cases} 0 & \text{if } j < q, \\ 1 & \text{if } j = q, \end{cases}$$

$$a_j(n+1) = \begin{cases} a_{j+1}(n) & \text{if } j < q, \\ 2a_1(n) + \sum_{1 \leq j \leq q} a_j(n) & \text{if } j = q. \end{cases}$$

It will be shown in 4.1 that  $a_j(n-1) = \nu'_j(n)$ .

**3.6.2. Proof of Proposition 3.8.** The proof of the proposition relies on the two lemmas below which loosely speaking say that the Königs coordinates  $\phi_h$  and its inverse  $\psi_h$  are close to the identity in  $\mathbb{D}$ , as  $r \rightarrow 1$ , since  $h$  converges to the identity in  $\mathbb{D}$ .

**Lemma 3.9.** *Let  $X$  be a branch of  $\overline{\mathcal{C}}_j$  at  $(\kappa, 0)$ . Given  $\varepsilon > 0$ , there exists  $0 < r_0 < 1$ ,  $C > 0$  and  $N \in \mathbb{N}$  such that if  $h = h_{r\kappa, \rho} \in X$  and  $r_0 < r < 1$ , then*

$$|\phi_h(z) - z| < \varepsilon|z|$$

for all  $|z| \leq 1 - C(1 - r)$  and,

$$|h^N(1 + \rho)| \leq 1 - C(1 - r).$$

*Proof.* Note that, there exists  $a \in \mathbb{C}$  with  $\Re a > 0$  such that, if  $h_{\zeta, \rho} \in X$ , then

$$\zeta = \kappa(1 + a\rho + o(\rho)),$$

as  $\rho \rightarrow 0$ . Assume that  $h_{r\kappa, \rho} \in X$  and  $r < 1$ . Let  $\Delta = 1 - r$ . It follows that

$$\rho = -\frac{\Delta}{a}(1 + o(\Delta))$$

as  $\Delta \rightarrow 0$ .

Given  $\varepsilon > 0$ , take

$$C \geq \frac{1}{2\varepsilon|a|^2} + \frac{1}{|a|}.$$

Assume that

$$|z| \leq 1 - C\Delta.$$

Then, for  $\Delta$  sufficiently small,

$$|1 + \frac{\rho}{2} - z| \geq -\frac{\Delta}{2|a|}(1 + o(\Delta)) + C\Delta \geq \frac{\Delta}{2\varepsilon|a|^2}.$$

Also recall that  $h_{\zeta, \rho}(z) = \zeta z P_\rho(z)$  where

$$P_\rho(z) = 1 - \frac{\rho^2 z}{4(1 + \rho)(1 + \rho/2 - z)}.$$

Thus, for  $\Delta$  sufficiently small,

$$|P_\rho(z) - 1| \leq \frac{(2\Delta/|a|)^2}{2\Delta/(2\varepsilon|a|^2)} \leq 4\varepsilon\Delta|z|.$$

Hence,

$$|h(z)| \leq (1 - \Delta)|z|(1 + |P_\rho(z) - 1|) \leq (1 - \Delta)|z|(1 + 4\varepsilon\Delta) \leq (1 - (1 - 4\varepsilon)\Delta)|z|.$$

In particular,  $|h(z)| \leq 1 - C\Delta$ .

Let  $z_n = h^n(z)$  and recall that  $\phi_h(z) = \lim \zeta^{-n} z_n$ .

$$\begin{aligned}
\left| \frac{z_{n+1}}{\zeta^{n+1}} - \frac{z_n}{\zeta^n} \right| &= (1 - \Delta)^{-n} |z_n| \cdot |P_\rho(z_n) - 1| \\
&\leq (1 - \Delta)^{-n} |z_n|^2 4\varepsilon \Delta \\
&\leq (1 - \Delta)^{-n} (1 - (1 - 4\varepsilon)\Delta)^{2n} |z_0| 4\varepsilon \Delta.
\end{aligned}$$

Hence,

$$|\zeta^{-(n+1)} z_{n+1} - z_0| \leq 4\varepsilon \Delta \sum \left( \frac{(1 - (1 - 4\varepsilon)\Delta)^2}{1 - \Delta} \right)^n |z_0| = \frac{4\varepsilon(1 - \Delta)}{1 - 8\varepsilon - (1 - 4\varepsilon)^2 \Delta} |z_0| \leq 8\varepsilon |z_0|$$

for  $\Delta$  sufficiently small and  $\varepsilon$  small, say  $\varepsilon < 1/20$ . This proves the first assertion.

For the second assertion, recall that as  $\rho \rightarrow 0$ ,

$$\frac{h^q(1 + z\rho) - 1}{\rho} \rightarrow g(z) = qa + z + \frac{1}{2(2z - 1)}$$

uniformly in compact subsets of  $\mathbb{C}$ . Since  $\infty$  is a parabolic fixed point of  $g$  with attracting direction  $(0, +\infty) \cdot a$ , it follows that, given  $\delta > 0$ , for  $N$  large,  $|g^N(1)|$  is large and

$$|\arg g^N(1) - \arg a| < \delta.$$

Since  $\rho = -\Delta(1 + o(1))/a$ , we have that

$$|1 + g^N(1)\rho| \leq 1 - (\cos \delta)|g^N(1)|\Delta.$$

Thus, taking  $\delta > 0$  small,  $N$  sufficiently large, and  $r$  sufficiently close to 1, we have that  $(h^{qN}(1 + \rho) - 1)/\rho$  is sufficiently close to a sufficiently large  $g^N(1)$  so that

$$|h^{qN}(1 + \rho)| \leq 1 - C\Delta.$$

□

**Lemma 3.10.** *Let  $X$  be a branch of  $\mathcal{C}_j$  at  $(\kappa, 0)$ . Given  $\varepsilon > 0$ , there exists  $0 < r_0 < 1$  and  $N' \in \mathbb{N}$  such that the following holds. For all  $\zeta = r\kappa$  with  $r_0 < r < 1$ , and all  $h = h_{r\kappa, \rho} \in X$  if*

$$w \in \{t\zeta^n \mid t \in ]0, 1]\},$$

*then*

$$|\psi_h(w) - w| < \varepsilon|w|,$$

*for all  $n \geq N'$ .*

*Proof.* Let  $\Delta = 1 - r$ . Consider  $N$  and  $C$  as in the previous lemma. Denote by  $\text{dist}_{\mathbb{D}}$  the hyperbolic distance (with constant curvature  $-1$ ) in the open unit disk. Observe that

$$\text{dist}_{\mathbb{D}}(\zeta^n, \zeta^{n+q}) = \log \frac{1 + r^n}{1 - r^n} \frac{1 - r^{n+q}}{1 + r^{n+q}} \leq q\Delta + \log 2.$$

Hence we may choose  $N' > N$  such that

$$\text{dist}_{\mathbb{D}}(\zeta^n, \zeta^{n+q}) < 1$$

for all  $n \geq N'$  and all  $\Delta$  sufficiently small.

We may assume that  $\Delta$  is sufficiently small so that the  $\varepsilon/2$  ball centered at  $\kappa^n$ , denoted  $B_{\varepsilon/2}(\kappa^n)$  is not contained in  $\psi_h(\mathbb{D})$ . In fact, since  $h$  converges uniformly in compacts subset of  $\mathbb{C} \setminus \{1\}$  to multiplication by  $\kappa$ , any such ball contains an iterated preimage of the critical point  $\omega_2 = 1$ .

We now claim that for all  $n \geq N'$ , if  $\psi_h(\zeta^n)$  and  $\psi_h(\zeta^{n+q})$  lie in  $B_{\varepsilon/2}(\kappa^n)$ , then the hyperbolic geodesic of  $\psi_h(\mathbb{D})$  joining these points is contained in  $B_\varepsilon(\kappa^n)$ . In fact, recall that the (infinitesimal) hyperbolic arc length in  $\psi_h(\mathbb{D})$  is bounded below by  $1/(2\delta(z))$  where  $\delta(z)$  is the distance from  $z$  to the boundary of  $\psi_h(\mathbb{D})$ . Thus the geodesic from  $\psi_h(\zeta^n)$  and  $\psi_h(\zeta^{n+q})$  may not exit  $B_\varepsilon(\kappa^n)$ , for otherwise, it would have length at least  $1/2$ .

By the previous lemma we may take  $\Delta$  sufficiently small so that the following two conditions are satisfied:

- For all  $w$  such that  $|w| \leq 1 - \varepsilon/5$ ,

$$|\psi_h(w) - w| < \varepsilon|w|.$$

- For all  $z$  such that  $|z| < 1 - C\Delta$  we have

$$|\phi_h(z) - z| < \varepsilon/8.$$

- For all  $n$  such that  $N' \leq n < N' + q$ , we may also assume that  $|\zeta^n - \kappa^n| < \varepsilon/8$ .

Recall that  $\psi_h(\zeta^k) = h^k(\omega_1)$ , for all  $k \geq 1$ . Equivalently,  $\phi_h(h^k(\omega_1)) = \zeta^k$ . We may assume that  $n$  is such that  $N' \leq n < N' + q$ , it follows that,

$$|h^n(\omega_1) - \kappa^n| \leq |h^n(\omega_1) - \zeta^n| + |\zeta^n - \kappa^n| < \varepsilon/2.$$

Let  $m \geq 1$  be such that

$$|h^{n+qm}(\omega_1) - \kappa^n| \leq \varepsilon/2$$

and

$$|h^{n+q(m+1)}(\omega_1) - \kappa^n| > \varepsilon/2.$$

Note that this number  $m$  depends on  $\Delta$  (that is on  $\zeta$ ).

It follows that

$$|\zeta^{n+q(m+1)} - \kappa^n| > \varepsilon/2 - \varepsilon/8.$$

Taking  $\Delta$  smaller if necessary, so that the second inequality below holds we have:

$$|\zeta^{n+q(m+1)} - \zeta^{n+qm}| \leq |\zeta^q - 1| < \varepsilon/8.$$

Hence,

$$|\zeta^{n+qm} - \kappa^n| > \varepsilon/2 - \varepsilon/8 - \varepsilon/8 > \varepsilon/5.$$

If  $w = s\zeta^n$ , for some  $s < r^{n+qm}$  then  $|w| \leq 1 - \varepsilon/5$  and therefore  $|\psi_h(w) - w| \leq \varepsilon|w|$ .

Finally, if  $w = s\zeta^n$  where  $r^{n+qj} \leq s \leq r^{n+q(j-1)}$  for some  $1 \leq j \leq m$ , then  $\psi_h(w)$  lies in a geodesic joining the points  $h^{n+q(j-1)}(\omega_1)$  and  $h^{n+qj}(\omega_1)$ , which are  $\varepsilon/2$ -close to  $\kappa^n$ . Thus,  $\psi_h(w)$  is  $\varepsilon$ -close to  $\kappa^n$ . For  $\Delta$  sufficiently small, it follows that  $|\psi_h(w) - w| \leq 3\varepsilon|w|$ .  $\square$

*Proof of Proposition 3.8.* It is sufficient to show that  $h(\omega_1) \in \gamma'_1$  for all  $r$  sufficiently close to 1.

Again we let  $\Delta = 1 - r$  and assume let  $\varepsilon > 0$  be sufficiently small so that the sectors

$$S_i = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z - \arg \kappa^i| < 3\varepsilon\}$$

for  $i = 0, \dots, q-1$  (subscripts mod  $q$ ) are pairwise disjoint.

For  $\Delta$  sufficiently small and  $N'$  as in the previous lemma, for all  $i = 0, \dots, q-1$ , we have that  $\psi_h([0, 1]\zeta^{N'+i}) \subset S_i$ . We will show that  $\gamma'_i \subset S_i$ . Since  $h(\omega_1) \in S_1$  the proposition will follow.

Let  $T_0 = \cup \psi_h([0, 1]\zeta^{N'+i})$  and note that  $h^M(\Gamma'_h) \subset T_0$  for  $M = N' + q + j$ .

We claim that

$$\Gamma'_h \subset \bar{S} = \bar{S}_0 \cup \cdots \cup S_{q-1}^-.$$

In order to prove this statement, given  $\delta > 0$ , let us denote by  $V_\delta$  a  $\delta$ -open neighborhood of the roots of unity  $1, \kappa, \dots, \kappa^{q-1}$ . Since  $h^M$  converges uniformly (spherical metric) to multiplication by  $\kappa^M$  in  $\bar{\mathbb{C}} \setminus V_\varepsilon$  as  $\Delta \rightarrow 0$ , we have that for  $\Delta$  sufficiently small, if  $z \in \bar{\mathbb{C}} \setminus V_\varepsilon$  and  $h^M(z) \in T_0 \cap V_{2\varepsilon}$ , then  $z \in T_0$ . We may also assume that if  $z \in \bar{\mathbb{C}} \setminus V_\varepsilon$  and  $h^M(z) \in V_{2\varepsilon}$ , then  $z \in V_{3\varepsilon}$ . Thus, given  $z \in \Gamma'_h$ , then  $h^M(z) \in T_0$  and therefore,  $z \in V_{3\varepsilon} \cup T_0$ .  $\square$

**3.7. Proof of Theorem 3.1.** By Bezout's Theorem, and our study of intersections at infinity, we have that  $\eta_{II}(m, j)$  is obtained by subtracting from product of the degrees of  $\bar{\mathcal{P}}_j$  and  $\bar{\mathcal{Q}}_{m-j}$  (first line below) the intersection numbers at  $[0 : 0 : 1]$  (second line), at  $[0 : 1 : 0]$  (third line), and at all  $[c_{p,q} : 0 : 1]$  (fourth line).

$$\begin{aligned} \eta_{II}(m, j) = & \left( \frac{1}{6}(2^j - (-1)^j) + \frac{1}{2} \right) \left( \frac{1}{6}(7 \cdot 2^{m-j} - (-1)^{m-j}) - \frac{1}{2} \right) \\ & - 2^{m-j-1} \\ & - \frac{1}{6}(1 - (-1)^j)(2^{m-j} - (-1)^{m-j}) \\ & - \frac{1}{2} \sum_{3 \leq q < j} \phi(q) \nu'_q(j) \nu'_q(m-j). \end{aligned}$$

Now a calculation shows that the formula above is equivalent to that stated in the theorem.  $\square$

#### 4. APPENDIX

The numbers  $\nu'_q(j)$  are characterized as follows.

**Lemma 4.1.** *Consider the  $q \times q$  matrix  $M = (M_{\ell, m})$  with non-negative integer entries given by:*

$$M_{\ell, m} = \begin{cases} 1 & \text{if } \ell < q \text{ and } m = \ell + 1 \\ 2 & \text{if } \ell = q \text{ and } m = 1 \\ 1 & \text{if } \ell = q \text{ and } m > 1 \\ 0 & \text{otherwise} \end{cases}$$

Let  $a_1(0) = \cdots = a_{q-1}(0) = 0$ ,  $a_q(0) = 1$  and for all integers  $n \geq 1$ ,

$$(a_1(n), \dots, a_q(n)) = M^n(a_1(0), \dots, a_q(0)).$$

Then, for all integers  $j \geq 0$ ,

$$\nu'_q(j+1) = a_1(j).$$

*Proof.* Let  $s(n) = a_1(n) + \cdots + a_q(n)$ . It follows that  $s(n+1) = 2s(n)$ . Thus  $s(n) = 2^n$ . Moreover, given  $0 \leq r < q$ , since  $a_1(kq+r) = a_q((k-1)q+r+1) = a_1((k-1)q+r) + s((k-1)q+r)$ , we obtain that

$$a_1(kq+r) = a_1(r) + s(r+q) + \cdots + s((k-1)q+r) = a_{1+r}(0) + s(r+q) + \cdots + s((k-1)q+r)$$

$$= \begin{cases} \frac{2^j - 2^r}{2^q - 1} & \text{if } r < q-1, \\ 1 + \frac{2^j - 2^r}{2^q - 1} & \text{if } r = q-1. \end{cases}$$

□

**Proposition 4.2.**

$$\sum_{3 \leq q < j} \phi(q) \nu'_q(j) = \frac{1}{3}(2^j + (-1)^j) - 1$$

**Lemma 4.3.** *Let  $q_c(z) = z^2 + c$  be the quadratic family. If  $q_c$  has no attracting fixed point, denote by  $\alpha_c$  the  $\alpha$  fixed point and let  $\alpha'_c = -\alpha_c$ . Given  $q \geq 2$  and  $p$  relatively prime to  $q$ , the number of parameters  $c$  in the  $p/q$  limb of the Mandelbrot set such that  $q_c^{j-1}(c) = \alpha'_c$  is  $\nu'_q(j)$ .*

*Proof.* We use the theory of dynamical rays and parameter rays developed by Douady and Hubbard [3]. These rays are the images of radii in the unit disc, under the inverse of the uniformising map onto, respectively the complement of the Julia set of  $q_c$ , and the complement of the Mandelbrot set. It is shown in ([3, Chapters V11, XIII], [12]) that dynamical or parameter rays of rational argument always land, and of course the landing point must be either preperiodic (for a dynamical ray) or of special type (for a parameter ray). Each of  $\alpha_c$  and  $\alpha'_c$  is the endpoint of exactly  $q$  rays of rational argument, which are the same throughout the  $p/q$  limb. Each parameter value  $c$  in the  $p/q$  limb for which  $q_c^{j-1}(c) = \alpha'_c$  is the endpoint of  $q$  rays of rational argument, which are the same as the dynamical rays for  $q_c$  which end at  $c \in q_c^{1-j}(\alpha'_c)$ , and hence these arguments map in  $j-1$  iterates under  $x \mapsto 2x \bmod 1$  to the arguments of  $\alpha'_c$ . Each such rational argument occurs precisely once. The number of  $j-1$ 'th preimages under  $x \mapsto 2x \bmod 1$  in the limb of a ray landing at  $\alpha'_c$  is the same as the number of  $j-1$ 'th preimages in the limb of  $\alpha'_c$  under  $q_c$ . So the number of  $c$  in the  $p/q$  limb with  $q_c^{j-1}(c) = \alpha'_c$  is the same as the number of points of  $q_d^{1-j}(\alpha'_d)$  in the dynamical  $(p/q)$  limb, for any  $d$  in the parameter  $p/q$  limb. Using the natural Markov partition of  $J(q_d)$  given by components of  $J(q_d) \setminus \{\alpha_d\}$ , this number is precisely  $a_1(j-1) = \nu'(j)$ , for  $a_1(j-1)$  as in 4.1. □

*Proof of Proposition 4.2.* Let

$$s(j) = \sum_{2 \leq q < j} \phi(q) \nu'_q(j).$$

Also consider

$$T(j) = \sum_{2 \leq i \leq j} s(i).$$

Note that  $T(j)$  is the total number of parameters  $c$  such that  $q_c^j(c) = \alpha_c$ . Such parameters are solutions of the degree  $2^{j+1}$  equation

$$q_c^{j+1}(c) - q_c^j(c) = 0$$

which has a root of multiplicity  $j+2$  at  $c=0$  and all other roots are simple (since they correspond to quadratic polynomials with the critical point eventually mapping to a fixed point). The solutions consists of the union of the solutions of  $q_c^j(c) = \beta_c$ ,  $q_c^j(c) = \alpha_c$  and  $c=0$ . The solutions of the first equation are in one to one correspondence with the arguments  $\theta \in \mathbb{R}/\mathbb{Z} \setminus \{0\}$  such that  $2^j\theta = 0$ , since the landing point of a Mandelbrot ray with such an angle  $\theta$  is a solution of  $q_c^j(c) = \beta_c$  and there is exactly one such ray landing at each solution. We conclude that

$$T(j) = 2^{j+1} - (j+2) - (2^j - 1) = 2^j - j - 1.$$



Hence,

$$s(j) = T(j) - T(j-1) = 2^{j-1} - 1.$$

An easy induction on the definition of  $\nu'_2(j)$  shows that

$$\nu'_2(j) = \frac{1}{3}(2^{j-1} - (-1)^{j-1}),$$

and the proposition follows.  $\square$

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